HOMFLY and superpolynomials for figure eight knot in all symmetric and antisymmetric representations

H. Itoyama[§], **A. Mironov**[¶], **A. Morozov**[‖], **And. Morozov**[∗∗]

Explicit answer is given for the HOMFLY polynomial of the figure eight knot 4_1 in arbitrary symmetric representation $R = [p]$. It generalizes the old answers for $p = 1$ and 2 and the recently derived results for $p = 3, 4$, which are fully consistent with the Ooguri-Vafa conjecture. The answer can be considered as a quantization of the $\mathfrak{H}_R = \mathfrak{H}_{[1]}^{|R|}$ identity for the "special" polynomials (they define the leading asymptotics of HOMFLY at $q = 1$), and arises in a form, convenient for comparison with the representation of the Jones polynomials as sums of dilogarithm ratios. In particular, we construct a difference equation ("non-commutative $\mathcal{A}$-polynomial") in the representation variable p. Simple symmetry transformation provides also a formula for arbitrary antisymmetric (fundamental) representation $R = [1^p]$, which also passes some obvious checks. Also straightforward is a deformation from HOMFLY to superpolynomials. Further generalizations seem possible to arbitrary Young diagrams R, but these expressions are harder to test because of the lack of alternative results, even partial.

1 Introduction

Recently a program was originated for a systematic study of knot polynomials [1, 2, 3, 4, 5] as functions of all their parameters, including the dependencies on representations and on knots. This is a very important problem, given the relevance of Chern-Simons theory [6] and its non-trivial deformations for future development and extension of our knowledge about conformal, topological and integrable systems, matrix models, instanton calculus and all kinds of dualities between them implied by the gauge/string correspondence, such as the AdS/CFT and AGT relations. Recent years the progress was dramatically slowed down by the lack of systematic description of correlators in Chern-Simons theory, comparable with the theory of conformal blocks, Nekrasov functions, Virasoro constraints, Sieberg-Witten equations and Dijkgraaf-Vafa phases. The complication here is appearance of new parameters, like knot topology and representation, which considerably enlarge the space of observables and poses new combinatorial problems. The main ingredients of our approach is the search for formulas, which explicitly depend on at least one parameter (in addition to the usual A, q, t, which are rather *arguments* of the knot polynomials), and emphasize on the set of methods developed in adjacent fields: character calculus, Ward identities and integrability properties.

The four keystone choices, which have just inspired a fast progress in this field are:

- Families of knots or representations instead of isolated examples (a given knot and a given representation)

- Braids instead of knots

[§]*Department of Mathematics and Physics, Osaka City University* and *Osaka City University Advanced Mathematical Institute (OCAMI), Osaka, Japan*; itoyama@sci.osaka-cu.ac.jp

[¶]*Lebedev Physics Institute* and *ITEP, Moscow, Russia*; mironov@itep.ru; mironov@lpi.ru

[‖]*ITEP, Moscow, Russia*; morozov@itep.ru

[∗∗]*Moscow State University* and *ITEP, Moscow, Russia*; Andrey.Morozov@itep.ru

- Braid group ($\mathcal{R}$-matrix) representation [7, 15] instead of skein relations [8]

- Character expansions instead of other group theory methods (broadly applied to the problem in the past decades).

This approach, though still a piece of art rather than a fully deductive method, already proved its power in the study of

- torus superpolynomials [9, 10] (dependence on the β-deformation [11] parameter)

- generic $m \leq 5$-strand braids in the fundamental representation [12, 13] (dependence on the knot)

- extension of knot polynomials to τ-function like quantities [14] (introduction and dependence on the time-variables)

The remaining direction, the dependence on representation variable, was well studied in the "classical" period of knot theory in the case of Jones polynomials, but only partial results exist for anything else except for a remarkable progress made by the Indian group [15, 16] (see also [17]). This direction is especially interesting because of the hidden Ward identities, a small piece of which were discovered in [18], and existing evidence [19, 20] of applicability of the AMM/EO recursion [21], which allows one to identify the representation variables with (some discretized and multiparametric!) version of the genus-expansion parameter. Very recently there was a breakthrough in the two cases (underlined are the free parameters):

- Arbitrary symmetric representation for the β-deformed Jones superpolynomial for the trefoil and other 2-strand knots (all torus) [22]: $\mathcal{K} = [2, \underline{2k+1}]$, $R = [\underline{p}]$, $A = t^2$

- Symmetric representation for arbitrary 3-strand HOMFLY polynomial (not obligatory torus) [23]: $\mathcal{K} = \underline{(a_1, b_1 | a_2, b_2 | \ldots)}$, $R = [2]$, $t = q$

In this paper we add one more 1-parametric result:

- $\boxed{\text{Arbitrary symmetric representation for the HOMFLY polynomial of the figure eight knot}}$:

  the simplest non-torus knot, which is 3-strand: $\mathcal{K} = 4_1 = (1, -1 | 1, -1)$, $R = [\underline{p}]$, $t = q$.
In fact, this immediately implies also the answer for all fully antisymmetric (i.e. all fundamental) representations, $R = [1^p]$.

The notation corresponds to the now standard parametrization of extended knot superpolynomials [14]: $\mathcal{P}_R^{\mathcal{B}}\{p_k | q, t\}$, with $t = q^\beta$. For $t = q$ (i.e. $\beta = 1$) this reduces to the extended HOMFLY polynomial $\mathcal{H}_R^{\mathcal{B}}\{p_k | q\} = \mathcal{P}_R^{\mathcal{B}}\{p_k | q, t = q\}$, which, if further reduced to a one-dimensional "topological locus" in the infinite-dimensional space of time-variables,

$$p_k = p_k^* = \frac{A^k - A^{-k}}{q^k - q^{-k}} \tag{1}$$

becomes the ordinary HOMFLY polynomial ${}^*\mathcal{H}_R^{\mathcal{K}}\{A | q\} = \mathcal{P}_R^{\mathcal{B}}\{p_k^* | q, t = q\}$, which does *not* depend on the braid representation $\mathcal{B}$ of the knot $\mathcal{K}$ and, after a further restriction $A = q^N$, possesses a QFT representation as an average of the Wilson line in $3d$ Chern-Simons theory:

$$ {}^*\mathcal{H}_R^{\mathcal{K}}\{A = q^N | q\} = \left\langle \mathrm{Tr}_R P \exp\left(\oint_{\mathcal{K}} \mathcal{A}\right) \right\rangle_{CS} \tag{2}$$

with the action $\frac{\kappa}{4\pi} \mathrm{Tr}\left(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right)$, the gauge group $SL(N)$ and $q = \exp\left(\frac{2\pi i}{\kappa + N}\right)$.

For a given number m of strands, the braid is parameterized by a semi-infinite matrix of integers

$$\mathcal{B} = (a_{11}, a_{12}, \ldots, a_{1,m-1} | a_{21}, a_{22}, \ldots, a_{2,m-1} | a_{31}, \ldots) \tag{3}$$

(if we go along the braid, at first, the first and the second strands intertwine a_{11} times, then, the second and the third intertwine a_{21} times and so on). In the 3-strand case we also denote $a_{1i} = a_i$, $a_{2i} = b_i$. For most such matrices the knots are either *composite* so that the corresponding HOMFLY polynomials are reducible, or they are actually not knots but links. The HOMFLY polynomials are irreducible for the *prime* knots, the first few (with $\sum_{ij} a_{ij} \leq 10$) listed in the Rolfsen table at [24].

There are further important reductions of the HOMFLY polynomials:

- to Alexander polynomials: $A = 1$
- to Jones polynomials: $A = q^2$
- to "special" polynomials: $q \to 1$.

2 The main result

Our claim in this paper is that

$$\boxed{{}^*\mathcal{H}^{4_1}_{[p]}(A|\,q) = {}^*S_{[p]}(A|\,q)\left(1 + \sum_{k=1}^{p} \frac{[p]!}{[k]![p-k]!}\prod_{i=0}^{k-1}\{Aq^{p+i}\}\{Aq^{i-1}\}\right)} \tag{4}$$

where $\{x\} = x - x^{-1}$ and $[x] = \frac{\{q^x\}}{\{q\}} = \frac{q^x - q^{-x}}{q - q^{-1}}$ is the "quantum number" and the "figure eight knot" $4_1 = (1,-1|1,-1)$ is the first non-torus knot, which possesses a 3-strand braid representation. Finally

$$ {}^*S_{[p]}(A|\,q) = S_{[p]}\{p_k^*\} = \prod_{i=1}^{p}\frac{\{Aq^{i-1}\}}{\{q^i\}} \tag{5}$$

are the symmetric Schur functions $S_R(p_k)$ taken at the special point in the space of time variables (1), at the topological locus. At this point, the Schur function is equal to the quantum dimension of the representation R and is equal to the value of the HOMFLY polynomial for the unknot (trivial knot), ${}^*\mathcal{H}^{\emptyset}_{[p]}(A|\,q) = {}^*S_{[p]}(A|\,q)$. For $A = q^N$ this is just a quantum dimension of the symmetric representation $R = [p]$ of the group $SL(N)$.

In particular, for the first three symmetric representations one explicitly has:

$$\frac{{}^*\mathcal{H}^{4_1}_{[0]}(A|\,q)}{{}^*S_{[0]}(A|\,q)} = 1,$$

$$\frac{{}^*\mathcal{H}^{4_1}_{[1]}(A|\,q)}{{}^*S_{[1]}(A|\,q)} = A^2 - q^2 + 1 - q^{-2} + A^{-2} = 1 + \{Aq\}\{Aq^{-1}\},$$

$$\frac{{}^*\mathcal{H}^{4_1}_{[2]}(A|\,q)}{{}^*S_{[2]}(A|\,q)} = q^4 A^4 - (q^6 + q^4 - q^2 + q^{-2})A^2 + (q^6 - q^4 + 3 - q^{-4} + q^{-6}) -$$

$$- (q^2 - q^{-2} + q^{-4} + q^{-6})A^{-2} + q^{-4}A^{-4} = 1 + [2]\{Aq^2\}\{Aq^{-1}\} + \{Aq^3\}\{Aq^2\}\{A\}\{Aq^{-1}\},$$

$$\frac{{}^*\mathcal{H}^{4_1}_{[3]}(A|\,q)}{{}^*S_{[3]}(A|\,q)} = 1 + [3]\{Aq^3\}\{Aq^{-1}\} + [3]\{Aq^4\}\{Aq^3\}\{A\}\{Aq^{-1}\} + \{Aq^5\}\{Aq^4\}\{Aq^3\}\{Aq\}\{A\}\{Aq^{-1}\}$$

$$\tag{6}$$

3 The answer for arbitrary antisymmetric representation

HOMFLY polynomials and their character expansions possess a Z_2-symmetry

$$A,\ q,\ {}^*S_R \quad \longleftrightarrow \quad A,\ -\frac{1}{q},\ {}^*S_{R'} \tag{7}$$

where R' is a transposition of the Young diagram R. It is important here that $S_{R'}\{p_k\} = S_R\{(-)^{k-1}p_k\}$. This symmetry allows to convert (4) for the single-row Young diagram $R = [p]$ into

3

an answer for its transposed counterpart, i.e. arbitrary single-column diagram $R = [1^p] = \underbrace{[1,1,\ldots,1]}_{p \text{ times}}$:

$$
{}^*\mathcal{H}^{4_1}_{[1^p]}(A\,|\,q) = {}^*S_{[1^p]}(A\,|\,q)\left(1 + \sum_{k=1}^{p}\frac{[p]!}{[k]![p-k]!}\prod_{j=0}^{k-1}\{Aq^{-p-j}\}\{Aq^{-j+1}\}\right)
\tag{8}
$$

It is easy to check that all the sign factors disappear from this expression. The relevant Schur functions at topological locus are given by

$$
{}^*S_{[1^p]}(A\,|\,q) = S_{[1^p]}\{p_k^*\} = \prod_{j=1}^{p}\frac{\{Aq^{1-j}\}}{\{q^j\}}
\tag{9}
$$

The first few examples of (8) are:

$$
\frac{{}^*\mathcal{H}^{4_1}_{[11]}(A\,|\,q)}{{}^*S_{[11]}(A\,|\,q)} = 1 + [2]\{Aq^{-2}\}\{Aq\} + \{Aq^{-3}\}\{Aq^{-2}\}\{A\}\{Aq\},
$$

$$
\frac{{}^*\mathcal{H}^{4_1}_{[111]}(A\,|\,q)}{{}^*S_{[111]}(A\,|\,q)} = 1 + [3]\{Aq^{-3}\}\{Aq^1\} + [3]\{Aq^{-4}\}\{Aq^{-3}\}\{A\}\{Aq\} + \{Aq^{-5}\}\{Aq^{-4}\}\{Aq^{-3}\}\{Aq^{-1}\}\{A\}\{Aq\}
\tag{10}
$$

It is easy to see that the r.h.s. of these expressions turn into unity at $A = q^2$, $A = q^3$ respectively. At the same time ${}^*S_{111}(A|q)$ also vanishes at $A = q^2$.

4 Comments on the shape of (4)

The answer (4) is, first of all, a straightforward generalization of the available examples (6) from [16, 23] (for $[p] = [2]$) and [25] (for $[p] = [3], [4]$).

The desire to look for the special shape of the answer, which is the key to generalization, comes from considering the "special" polynomials. The special polynomial [26] is obtained from HOMFLY polynomials at $q \to 1$:

$$
\mathfrak{H}^{\mathcal{K}}_R(A) = \lim_{q\to 1}\frac{{}^*\mathcal{H}^{\mathcal{K}}_R(A\,|\,q)}{{}^*S_R(A\,|\,q)}
\tag{11}
$$

Note that the limit is taken with fixed A, and both the HOMFLY polynomial ${}^*\mathcal{H}_R$ and the quantum dimension ${}^*S_R$ are singular behaving as $(q - q^{-1})^{-|R|}$. Here $|R|$ is the number of boxes in the Young diagram R, in particular, $||p|| = p$. The conjectured property of "special" polynomials reads as [9]

$$
\boxed{\mathfrak{H}^{\mathcal{K}}_R(A) = \left(\mathfrak{H}^{\mathcal{K}}_{[1]}(A)\right)^{|R|}}
\tag{12}
$$

and is presumably valid for arbitrary $\mathcal{K}$ and R. It is an amusing "dual" of a somewhat similar conjecture

$$
\boxed{\mathfrak{A}^{\mathcal{K}}_R(q) = \mathfrak{A}^{\mathcal{K}}_{[1]}\left(q^{|R|}\right)}
\tag{13}
$$

for the Alexander polynomial

$$
\mathfrak{A}^{\mathcal{K}}_R(q) = \lim_{A\to 1}\frac{{}^*\mathcal{H}^{\mathcal{K}}_R(A\,|\,q)}{{}^*S_R(A\,|\,q)}
\tag{14}
$$

in any symmetric or antisymmetric representations[1].

Since $\mathfrak{H}^{4_1}_{[1]} = 1 + \{A\}^2$, it follows from (12) that $\frac{{}^*\mathcal{H}^{4_1}_R(A\,|\,q)}{{}^*S_R(A\,|\,q)}$ is a quantization of

$$
\mathfrak{H}^{4_1}_R(A) = \left(1 + \{A\}^2\right)^{|R|} = \sum_{j=0}^{|R|}\frac{|R|!}{j!(|R|-j)!}\{A\}^{2j}
\tag{15}
$$

---

[1]This is also correct, at least, for the torus knots in all representations described by the corner Young diagrams.

This explains the shape of eq.(4).

Additional support for this kind of representation for the answer comes from the Pochhammer (dilogarithm) decompositions of the Jones polynomials.

The last piece of support is provided by the formula from [17] for $^*H^{4_1}_{[p]}(A|\,q)$ at the peculiar point $A = q^N$, $q = \exp\left(\frac{\pi i}{N+p-1}\right)$, which is easily reproduced from our (4). Indeed, the main formula of [17],

$$\frac{^*H^{4_1}_{[p]}}{^*S_{[p]}}\left(A = q^N \Big| q = e^{\frac{i\pi}{N+p-1}}\right) = \left(1 + \sum_{j=1}^{p}\prod_{i=0}^{j-1}\left(2\sin\frac{(p-i)\pi}{N+p-1}\right)^2\right) \tag{16}$$

is based on the same kind of decomposition, only the items are full squares, which is, indeed, the case at this special point and one can check that (16) follows from (4).

5 Evidence in favor of (4)

We have seven pieces of evidence:

- Eq.(4) reproduces (6)

- For $q \to 1$ eq.(4) reproduces the conjectured special polynomials

- Eq.(4) is consistent with the interesting formula (16) from [17], describing the value of $^*\mathcal{H}^{4_1}_{[p]}(A|q)$ at the one-dimensional locus $q = e^{\frac{i\pi}{N+p-1}}$, $A = q^N = -e^{\frac{i\pi(1-p)}{N+p-1}}$

- For $A = q^2$ eq.(4) reproduces the known answers for the Jones polynomials, moreover, directly in their most advanced form in terms of the dilogarithm ratios (products of the Pochhammer symbols)

- For $A = 1$ eq.(4) reproduces the Alexander polynomial, which depends on the representation R through (13)

- According to (8) the antisymmetric HOMFLY polynomial $^*\mathcal{H}^{4_1}_{[1^p]}(A|\,q)$ vanishes for $A = q^N$ with $N < p$, i.e. whenever p exceeds the rank of the group by two, and turns its ratio to the unknot into unity for $N = p$. Since (8) and (4) are related by the symmetry transform (7), this necessary property simultaneously checks (4)

- Eq.(4) is consistent with the Ooguri-Vafa conjecture

The first two points in this list are not independent checks, since they were used to write (4) down. However, the last five are. In what follows we briefly comment on the first and the last stories in this list. Other points do not seem to require additional explanations.

6 On the derivation of (6)

The key source of intuition for the quantization rule (15) $\longrightarrow$ (4) are explicit formulas (6). They are obtained from the character expansion of the extended HOMFLY polynomials for a generic 3-strand braid [14, 12, 23, 25]:

$$\mathcal{H}^{(a_1,b_1|a_2,b_2|...)}_R\{p_k|q\} = \sum_{Q\in R^{\otimes m}} S_Q\{p_k\}\cdot \mathrm{Tr}_{N_{RQ}\otimes N_{RQ}}\left\{\hat{\mathcal{R}}^{a_1}_{RQ}\hat{U}\hat{\mathcal{R}}^{b_1}_{RQ}\hat{U}^\dagger_{RQ}\hat{\mathcal{R}}^{a_2}_{RQ}\hat{U}_{RQ}\hat{\mathcal{R}}^{b_2}_{RQ}\hat{U}^\dagger_{RQ}\cdots\right\} \tag{17}$$

at the topological locus (1). In this particular case $m = 3$, but equally simple and general expressions can be written for arbitrary m, see [14, 12]. The figure eight knot is characterized by $a_1 = a_2 = 1$, $b_1 = b_2 = -1$.

The sum here goes over the Young diagrams $Q \in R^{\otimes m}$, which appear N_{RQ} times in the decomposition of the m-the power of representation R and have the size $|Q| = m|R|$. The Schur functions $S_Q\{p_k\}$ are well known for all Q. The multiplicities N_{RQ} define the sizes of the quantum $\mathcal{R}$-matrices, projected onto the space of the intertwining operators, we call them $\hat{\mathcal{R}}$ and they are diagonal matrices of the form

$$\hat{\mathcal{R}} = \mathrm{diag}\left(\epsilon_i\, q^{\varkappa_i} \;\middle|\; i = 1, \ldots, N_{RQ} \right) \tag{18}$$

with certain exponents $\varkappa_i$ and signs $\epsilon_i = \pm$, depending on the choice of R and $Q \in R^{\otimes m}$. The mixing matrices $\hat{U}_{RQ}$ describe the transition between bases in the N_{RQ}-dimensional space of intertwiners $R^{\otimes m} \longrightarrow Q$, which diagonalize $\hat{\mathcal{R}}$-matrices, associated with the crossings of the strands 12 and 23 in the braid (for $m > 3$ there will be $m - 2$ different mixing matrices for given R and Q). In practice, for $m = 3$ these matrices depend only on N_{RQ} and they are concrete functions of the parameters $\{\epsilon_i, \varkappa_i\}$ in the corresponding $\hat{\mathcal{R}}_{RQ}$. These functions are explicitly found in [25].[2] Therefore

$$^*\mathcal{H}^{4_1}_{[3]}(A|q) = {}^*S_{[9]}(A|q) + \left(q^{12} - 2q^6 + 1 - 2q^{-6} + q^{-12}\right){}^*S_{[81]}(A|q) +$$

$$+\left(q^{20} - 2q^{16} - 2q^{14} + q^{12} + 4q^{10} + q^8 - 4q^6 - 4q^4 + 2q^2 + 6 + 2q^{-2} - 4q^{-4} - 4q^{-6} + q^{-8} + 4q^{-10} + q^{-12} - 2q^{-14} - 2q^{-16} + q^{-20}\right){}^*S_{[72]}(A|q) +$$

$$+{}^*S_{[711]}(A|q) + \underline{\alpha_{[63]}\,{}^*S_{[63]}(A|q)} + \left(q^8 - 2q^4 + 1 - 2q^{-4} + q^{-8}\right){}^*S_{[621]}(A|q) + \left(q^8 - 2q^4 + 1 - 2q^{-4} + q^{-8}\right){}^*S_{[54]}(A|q) +$$

$$+\left(q^{12} - 2q^{10} - q^8 + 4q^6 - 3q^4 - 2q^2 + 6 - 2q^{-2} - 3q^{-4} + 4q^{-6} - q^{-8} - 2q^{-10} + q^{-12}\right){}^*S_{[531]}(A|q) +$$

$$+{}^*S_{[522]}(A|\,q) + {}^*S_{[441]}(A|q) + \left(q^4 - 2q^2 + 1 - 2q^{-2} + q^{-4}\right){}^*S_{[432]}(A|q) + {}^*S_{[333]}(A|q)$$

where $\alpha_{[63]}$ in the underlined term is given by a trace of the 4×4 matrix and is equal to

$$\alpha_{[63]} = q^{24} - 2q^{22} - q^{20} + 2q^{18} + 3q^{16} - 2q^{14} - 6q^{12} + 4q^{10} + 8q^8 - 2q^6 - 12q^4 + 2q^2 + 11 +$$

$$+2q^{-2} - 12q^{-4} - 2q^{-6} + 8q^{-8} + 4q^{-10} - 6q^{-12} - 2q^{-14} + 3q^{-16} + 2q^{-18} - q^{-20} - 2q^{-22} + q^{-24}$$

This formula reproduces the known answers for the colored Jones polynomial ($A = q^2$) from [24][3] and after the substitution of explicit expressions for $^*S_Q(A|q)$ it coincides with (4).

For $R = [4]$ there will be two items with 4×4 matrices and one item with a 5×5 matrix:

$$^*\mathcal{H}^{4_1}_{[4]}(A|\,q) = {}^*S_{[12]}(A|\,q) + \left(q^{16} - 2q^8 + 1 - 2q^{-8} + q^{-16}\right){}^*S_{[11,1]}(A|\,q) + \left(q^{28} - 2q^{22} - 2q^{20} + q^{16} + 4q^{14} + q^{12} - \right.$$

$$\left. - 4q^8 - 4q^6 + 2q^2 + 6 + 2q^{-2} - 4q^{-6} - 4q^{-8} + q^{-12} + 4q^{-14} + q^{-16} - 2q^{-20} - 2q^{-22} + q^{-28}\right){}^*S_{[10,2]}(A|\,q) +$$

$$+{}^*S_{[10,1,1]}(A|\,q) + \underline{\alpha_{[9,3]}\,{}^*S_{[9,3]}(A|\,q)} + \left(q^{12} - 2q^6 + 1 - 2q^{-6} + q^{-12}\right){}^*S_{[9,2,1]}(A|\,q) + \underline{\alpha_{[8,4]}\,{}^*S_{[8,4]}(A|\,q)} +$$

$$+\left(q^{20} - 2q^{16} - 2q^{14} + q^{12} + 4q^{10} + q^8 - 4q^6 - 4q^4 + 2q^2 + 6 + 2q^{-2} - 4q^{-4} - 4q^{-6} + q^{-8} + 4q^{-10} + q^{-12} - \right.$$

$$\left. - 2q^{-14} - 2q^{-16} + q^{-20}\right)\left({}^*S_{[8,3,1]}(A|\,q) + {}^*S_{[7,5]}(A|\,q)\right) + {}^*S_{[8,2,2]}(A|\,q) + \underline{\alpha_{[7,4,1]}\,{}^*S_{[7,4,1]}(A|\,q)} +$$

$$+\left(q^8 - 2q^4 + 1 - 2q^{-4} + q^{-8}\right)\left({}^*S_{[7,3,2]}(A|\,q) + {}^*S_{[6,5,1]}(A|\,q)\right) + {}^*S_{[6,6]}(A|\,q) +$$

$$+\left(q^{12} - 2q^{10} - q^8 + 4q^6 - 3q^4 - 2q^2 + 6 - 2q^{-2} - 3q^{-4} + 4q^{-6} - q^{-8} - 2q^{-10} + q^{-12}\right){}^*S_{[6,4,2]}(A|\,q) +$$

$$+{}^*S_{[6,3,3]}(A|\,q) + {}^*S_{[5,5,2]}(A|\,q) + \left(q^4 - 2q^2 + 1 - 2q^{-2} + q^{-4}\right){}^*S_{[5,4,3]}(A|\,q) + {}^*S_{[4,4,4]}(A|\,q)$$

---

[2] For $N_{RQ} \leq 3$ this result has been long ago obtained by a slightly different method in the beautiful paper [16]. Some other particular cases can be obtained using the formulas for the $SU_q(2)$ Racah coefficients, [27]. This idea goes back to [28] and was successfully developed in [15].

[3] The answer is also consistent with the Alexander polynomial ($A = 1$) in [24], but $^*S_{[63]}(A|\,q)$ vanishes at $A = 1$ and, therefore, this comparison is insensitive to the value $\alpha_{[63]}$. Similarly, comparison with the "special" polynomial $\mathfrak{H}^{4_1}_{[3]}(A)$ allows one only to check the value of $\alpha_{[63]}$ at $q = 1$.

with

$$\alpha_{[9,3]} = q^{36} - 2q^{32} - 2q^{30} - q^{28} + 4q^{26} + 5q^{24} + 2q^{22} - 5q^{20} - 10q^{18} - 3q^{16} + 8q^{14} + 14q^{12} +$$
$$+6q^{10} - 9q^8 - 18q^6 - 9q^4 + 10q^2 + 19 + 10q^{-2} - 9q^{-4} - 18q^{-6} - 9q^{-8} + 6q^{-10} +$$
$$+14q^{-12} + 8q^{-14} - 3q^{-16} - 10q^{-18} - 5q^{-20} + 2q^{-22} + 5q^{-24} + 4q^{-26} - q^{-28} - 2q^{-30} - 2q^{-32} + q^{-36}$$

$$\alpha_{[8,4]} = q^{40} - 2q^{38} - q^{36} + 2q^{34} + q^{32} + 4q^{30} - 6q^{28} - 4q^{26} + 2q^{24} + 4q^{22} + 13q^{20} - 10q^{18} - 15q^{16} - 4q^{14} +$$
$$+10q^{12} + 28q^{10} - 6q^8 - 24q^6 - 18q^4 + 6q^2 + 37 + 6q^{-2} - 18q^{-4} - 24q^{-6} - 6q^{-8} + 28q^{-10} + 10q^{-12} - 4q^{-14} -$$
$$-15q^{-16} - 10q^{-18} + 13q^{-20} + 4q^{-22} + 2q^{-24} - 4q^{-26} - 6q^{-28} + 4q^{-30} + q^{-32} + 2q^{-34} - q^{-36} - 2q^{-38} + q^{-40}$$

$$\alpha_{[7,4,1]} = q^{24} - 2q^{22} - q^{20} + 2q^{18} + 3q^{16} - 2q^{14} - 6q^{12} + 4q^{10} + 8q^8 - 2q^6 - 12q^4 + 2q^2 + 11 +$$
$$+2q^{-2} - 12q^{-4} - 2q^{-6} + 8q^{-8} + 4q^{-10} - 6q^{-12} - 2q^{-14} + 3q^{-16} + 2q^{-18} - q^{-20} - 2q^{-22} + q^{-24}$$

Again, this expression reproduces the known Jones and Alexander polynomials and again, Alexander is not sensitive to concrete values of the three α-parameters. Remarkably, these huge formulas, the best what can be directly deduced from the braid calculus, can be encapsulated into a simple expression (4) on the topological locus (1).

To conclude this section, we note that the character expansion formulas like (17) are very different in structure from the expansions like (4), implied by the quantization of the "special" polynomial basic identity (15), which is, in turn, closely related to the Pochhammer decompositions of [29]. It will be extremely important to understand the relation (duality?) between such decompositions in full generality.

7   Check of the Ooguri-Vafa conjecture

The HOMFLY polynomials are rather huge. It is also possible to express them through somewhat different Ooguri-Vafa (OV) polynomials [30]. At least, in the case of the figure eight knot the OV polynomials are partly factorizable and, hence, look much simpler than the original HOMFLY polynomials. The relation between the two involves a relation between the disconnected and connected correlators and is given by comparing the generating functions of all representations

$$\log\left(\sum_R \mathcal{H}_R^{\mathcal{K}}\{p_k|q\}S_R\{\bar{p}_k\}\right) = \sum_n \sum_R \frac{1}{n}\, f_R^{\mathcal{K}}\{p_k^{(n)}|q^n\}S_R\{\bar{p}_k^{(n)}\} \qquad (19)$$

and at the topological locus (1) this reduces to

$$\log\left(\sum_R {}^*\mathcal{H}_R^{\mathcal{K}}(A|q)S_R\{\bar{p}_k\}\right) = \sum_n \sum_R \frac{1}{n}\, {}^*f_R^{\mathcal{K}}(A^n|q^n)S_R\{\bar{p}_k^{(n)}\} \qquad (20)$$

Here $\{\bar{p}_k\}$ is an additional set of arbitrary time variables, and $\bar{p}_k^{(n)} \equiv \bar{p}_{nk}$ is their Adams transform. The OV conjecture claims [30] that, like the ratios ${}^*\mathcal{H}_R/{}^*S_R$, the ratio ${}^*f_R/{}^*S_{[1]}$ is always a polynomial with integer coefficients. This means that ${}^*f_R$ is always proportional to ${}^*S_{[1]}$, i.e. is less singular then ${}^*\mathcal{H}_R$, in the ("special-polynomial") limit of $q \to 1$ with A fixed.

The first few OV polynomials are

$$f_{[1]} = \mathcal{H}_{[1]}$$

$$f_{[2]} = \mathcal{H}_{[2]} - \frac{1}{2}\mathcal{H}_{[1]}^2 - \frac{1}{2}\mathcal{H}_{[1]}^{(2)}$$

$$f_{[3]} = \mathcal{H}_{[3]} - \mathcal{H}_{[1]}\mathcal{H}_{[2]} + \frac{1}{3}\mathcal{H}_{[1]}^3 - \frac{1}{3}\mathcal{H}_{[1]}^{(3)}$$

$$f_{[4]} = \mathcal{H}_{[4]} - \mathcal{H}_{[1]}\mathcal{H}_{[3]} - \frac{1}{2}\mathcal{H}_{[2]}^2 + \mathcal{H}_{[1]}^2\mathcal{H}_{[2]} - \frac{1}{4}\mathcal{H}_{[1]}^4 + \frac{1}{2}\mathcal{H}_{[2]}^{(2)} + \frac{1}{4}\left(\mathcal{H}_{[1]}^{(2)}\right)^2$$

(21)

where $\mathcal{H}_R^{(n)}\{p_k|q\}) \equiv \mathcal{H}_R\{p_k^{(n)}|q^n\}$ and $^*\mathcal{H}_R^{(n)}(A|q) \equiv {}^*\mathcal{H}_R(A^n|q^n)$.

In the examples (6) one has:

$$\frac{^*f_{[0]}^{4_1}(A|q)}{^*S_{[1]}(A|q)} = 0$$

$$\frac{^*f_{[1]}^{4_1}(A|q)}{^*S_{[1]}(A|q)} = A^2 - (q^2 - 1 + q^{-2}) + A^{-2}$$

$$\frac{^*f_{[2]}^{4_1}(A|q)}{^*S_{[1]}(A|q)} = \{A\}\{A/q\}\{Aq^2\}\{A^2q^2\},$$

$$\frac{^*f_{[3]}^{4_1}(A|q)}{^*S_{[1]}(A|q)} = \{A\}\{A/q\}\{Aq\}\{Aq^2\}\Big((q^8 + q^4 + q^2)A^4 - (q^6 - q^4 - 1)A^2 - $$

$$-(q^2 - 2 + q^{-2}) - (q^{-6} - q^{-4} - 1)A^{-2} + (q^{-8} + q^{-4} + q^{-2})A^{-4}\Big)$$

$$\frac{^*f_{[4]}^{4_1}(A|q)}{^*S_{[1]}(A|q)} = \{A\}\{A/q\}\{Aq\}\{Aq^2\}\Big((q^{19} + q^{15} + q^{13} + 2q^{11} + q^9 + 2q^7 + q^5 + q^3)A^7 - $$

$$-(q^{17} + q^{15} + q^{13} + 2q^{11} + 2q^9 + 2q^7 + q^5)A^5 + (q^{13} + q^9 - q^5 - q^3 - 2q^{-1})A^3 + $$

$$+(q^7 + q^5 + q^3 + 2q - q^{-5})A + (q^{-7} + q^{-5} + q^{-3} + 2q^{-1} - q^5)A^{-1} + $$

$$+(q^{-13} + q^{-9} - q^{-5} - q^{-3} - 2q^1)A^{-3} - (q^{-17} + q^{-15} + q^{-13} + 2q^{-11} + 2q^{-9} + 2q^{-7} + q^{-5})A^{-5} + $$

$$+(q^{-19} + q^{-15} + q^{-13} + 2q^{-11} + q^{-9} + 2q^{-7} + q^{-5} + q^{-3})A^{-7}\Big)$$

$$\dots$$

(22)

For these polynomials the Ooguri-Vafa conjecture is obviously true. It would be interesting to check it in full generality (for arbitrary $[p]$) for the answer (4).

In the special limit of $q \to 1$, the special OV polynomials $\mathfrak{f}_R(A) \equiv \lim_{q\to 1} \frac{^*f_R(A|q)}{^*S_{[1]}(A|q)}$ are

$$\mathfrak{f}_{[0]}(A) = 1,$$

$$\mathfrak{f}_{[1]}(A) = 1 + \{A\}^2,$$

$$\mathfrak{f}_{[2]}(A) = \{A\}^3\{A^2\},$$

$$\mathfrak{f}_{[3]}(A) = \{A\}^4\left(3A^4 + A^2 + A^{-2} + 3A^{-4}\right),$$

$$\mathfrak{f}_{[4]}(A) = 2\{A\}^3\{A^2\}\left(5A^6 - 10A^4 + 9A^2 - 7 + 9A^{-2} - 10A^{-4} + 5A^{-6}\right),$$

(23)

$$\dots$$

8

These polynomials non-trivially depend on representation, and can not be expressed through the special polynomials $\mathfrak{H}_R(A)$: higher order corrections in $1 - q$ also contribute to $\mathfrak{f}_{[R]}(A)$, see [26].

Likewise one can easily obtain from (8) the OV polynomials for the antisymmetric representations.

8 Colored superpolynomial for the figure eight knot

Our main formula (4) has, in fact, a very suggestive form, calling for various generalizations. However, for this purpose on should interpret it in a proper way. As it is, (4) looks like a sum over either the sub-diagrams of the Young diagram R, or simply the subsets of boxes in R. It turns our that the adequate one is the second representation. As we demonstrate in this section it provides an immediate suggestion for the *super*polynomial (β-) deformation of (4).

β-deformation implies that the shifts along the horizontal and vertical axes in Young diagram produce the q and t^{-1}, rather than q and q^{-1} factors in all the formulas. However, there is no immediate rule for the substitution (refinement) of the binomial coefficients. The β-deformation would be much simpler if sum over boxes would contain only *unit* coefficients. This means that in order to be straightforwardly deformed, (4) should first be rewritten in such a way.

This is, indeed, possible. To begin with,

$$[p]\{Aq^{p+i}\}\{Aa^{i-1}\} = \sum_{i=1}^{p}\{Aq^{2(p-i)+1}\}\{Aq^{-1}\} \tag{24}$$

i.e. this contribution can be immediately rewritten as a sum over boxes with unit coefficients. This implies that with every box with coordinates $(i, 1) \in [p]$ one associates a product $Z_i(A) = \{Aq^{2(p-i)+1}\}\{Aq^{-1}\}$. Next,

$$\frac{[p][p-1]}{[2]}\{Aq^{p+i+1}\}\{Aq^{p+i}\}\{A\}\{Aq^{-1}\} =$$

$$= \sum_{1 \leq i < i' \leq p}\{Aq^{2(p-i)+1}\}\{Aq^{-1}\}\{Aq^{2(p-i')+2}\}\{A\} = \sum_{1 \leq i < i' \leq p} Z_i(A)Z_{i'}(Aq) \tag{25}$$

i.e. the argument of the second factor $Z_{i'}$ is shifted by the factor of q. In general (4) can be rewritten as

$$\frac{{}^{*}\mathcal{H}_{[p]}^{4_1}(A\,|\,q)}{{}^{*}S_{[p]}(A\,|\,q)} = \sum_{k=0}^{p}\frac{[|R|]!}{[k]![|R|-k]!}\prod_{i=1}^{k}Z_i(A) = \sum_{k=0}^{p}\sum_{1 \leq i_1 < \ldots < i_k \leq p} Z_{i_1}(A)Z_{i_2}(Aq)Z_{i_3}(Aq^2)\ldots Z_{i_k}(Aq^{k-1}) \tag{26}$$

This formula already admits a straightforward β-deformation: it is enough to change $Z_i(A) = \{Aq^{2(p-i)+1}\}\{Aq^{-1}\}$ for $\mathfrak{Z}_i(A) = \{Aq^{2(p-i)+1}\}\{At^{-1}\}$, and

$$\boxed{\frac{{}^{*}\mathcal{P}_{[p]}^{4_1}(A\,|\,q,t)}{{}^{*}M_{[p]}(A\,|\,q,t)} = \sum_{k=0}^{p}\sum_{1 \leq i_1 < \ldots < i_k \leq p} \mathfrak{Z}_{i_1}(A)\mathfrak{Z}_{i_2}(Aq)\mathfrak{Z}_{i_3}(Aq^2)\ldots \mathfrak{Z}_{i_k}(Aq^{k-1})} \tag{27}$$

is the superpolynomial for knot 4_1 in arbitrary symmetric representation $[p]$. (Denominator at the l.h.s. is the value of the MacDonald polynomial at topological locus, which is the standard choice for the superpolynomial unknot, see [9] and references therein). The evidence in favor of this conjecture involves the two facts:

- For $t = q$ eq.(27) reproduces our HOMFLY polynomial (4).
- After the standard change of variables [5, 9],

$$t = \mathbf{q}, \qquad q = -\mathbf{q}\mathbf{t}, \qquad A = \mathbf{a}\sqrt{-\mathbf{t}} \tag{28}$$

the superpolynomial (27) is, indeed, a polynomial in all its variables $\mathbf{a}^{\pm 1}$, $\mathbf{q}^{\pm 1}$, $\mathbf{t}^{\pm 1}$ with all positive coefficients, simply because each factor $\mathcal{Z}_i(Aq^s)$ acquires this positivity property in the new variables:

$$\mathfrak{Z}_i(Aq^s) = \frac{\left(1 - A^2 q^{4(p-i)+2+2s}\right)\left(t^2 - A^2 q^{2s}\right)}{(A^2 tq) \cdot q^{2(p-i+s)}} = \frac{\left(1 + \mathbf{a}^2\mathbf{t}(\mathbf{qt})^{4(p-i)+2+2s}\right)\left(\mathbf{q}^2 + \mathbf{a}^2\mathbf{t}(\mathbf{qt})^{2s}\right)}{\mathbf{a}^2 \cdot (\mathbf{qt})^{2(p-i+s+1)}} \tag{29}$$

The symmetry (7) has an obvious generalization to the β-deformed situation (see also [22, 31]), just

$$(q,t) \longrightarrow (-t^{-1}, -q^{-1}) \qquad \text{or} \qquad (\mathbf{q},\mathbf{t}) \longrightarrow (1/\mathbf{qt}, \mathbf{t}) \tag{30}$$

which allows one to convert (27) into a similar formula for the superpolynomials in the fundamental representations:

$$\boxed{\frac{{}^*\mathcal{P}^{4_1}_{[1^p]}(A\,|\,q,t)}{{}^*M_{[1^p]}(A\,|\,q,t)} \;=\; \sum_{k=0}^{p}\; \sum_{1 \le i_1 < ... < i_k \le p} \bar{\mathfrak{Z}}_{i_1}(A)\bar{\mathfrak{Z}}_{i_2}(At^{-1})\bar{\mathfrak{Z}}_{i_3}(At^{-2})\dots\bar{\mathfrak{Z}}_{i_k}(At^{-k+1})} \tag{31}$$

with

$$\bar{\mathfrak{Z}}_i(At^{-s}) = \frac{\left(1 - A^2 t^{-4(p-i)-2-2s}\right)\left(q^{-2} - A^2 t^{-2s}\right)}{(A^2/tq) \cdot t^{-2(p-i+s)}} = \frac{\left(1 + \mathbf{a}^2\mathbf{tq}^{-4(p-i)-2-2s}\right)\left(1 + \mathbf{a}^2\mathbf{t}^3\mathbf{q}^{2(1-s)}\right)}{\mathbf{a}^2 \cdot \mathbf{t}^2\mathbf{q}^{-2(p-i+s)}} \tag{32}$$

which also possesses the positivity property in the bold variables (28). For illustration we provide just a few simplest examples of superpolynomials for the figure eight knot:

$$\frac{{}^*\mathcal{P}^{4_1}_{[1]}(A|q,t)}{{}^*M_{[1]}(A\,|\,q,t)} \;=\; 1 + \{Aq\}\{At^{-1}\} = 1 + \mathbf{t}^2\mathbf{a}^2 + \mathbf{q}^{-2}\mathbf{t}^{-1} + \mathbf{q}^2\mathbf{t} + \mathbf{t}^{-2}\mathbf{a}^{-2}$$

$$\frac{{}^*\mathcal{P}^{4_1}_{[2]}(A|q,t)}{{}^*M_{[2]}(A\,|\,q,t)} \;=\; 1 + \{Aq\}\{At^{-1}\} + \{Aq^3\}\{At^{-1}\} + \{Aq^3\}\{At^{-1}\}\{Aq^2\}\{Aqt^{-1}\} =$$

$$= \mathbf{a}^4\mathbf{q}^4\mathbf{t}^8 + \mathbf{a}^2(\mathbf{q}^{-2}\mathbf{t} + \mathbf{t}^2 + \mathbf{t}^3 + \mathbf{q}^2\mathbf{t}^4 + \mathbf{q}^4\mathbf{t}^5 + \mathbf{q}^6\mathbf{t}^7) + \mathbf{q}^6\mathbf{t}^4 + \mathbf{q}^4\mathbf{t}^3 + \mathbf{q}^2\mathbf{t}^2 + \mathbf{q}^2\mathbf{t} + 3 + \mathbf{q}^{-2}\mathbf{t}^{-1} + \mathbf{q}^{-2}\mathbf{t}^{-2} +$$

$$+ \mathbf{q}^{-4}\mathbf{t}^{-3} + \mathbf{q}^{-6}\mathbf{t}^{-4} + \mathbf{a}^{-2}(\mathbf{q}^2\mathbf{t}^{-1} + \mathbf{t}^{-2} + \mathbf{t}^{-3} + \mathbf{q}^{-2}\mathbf{t}^{-4} + \mathbf{q}^{-4}\mathbf{t}^{-5} + \mathbf{q}^{-6}\mathbf{t}^{-7}) + \mathbf{a}^{-4}\mathbf{q}^{-4}\mathbf{t}^{-8}$$

$$\frac{{}^*\mathcal{P}^{4_1}_{[11]}(A|q,t)}{{}^*M_{[11]}(A\,|\,q,t)} \;=\; 1 + \{Aq\}\{At^{-1}\} + \{Aq\}\{At^{-3}\} + \{Aq\}\{At^{-3}\}\{At^{-2}\}\{Aqt^{-1}\} =$$

$$= \mathbf{a}^4\mathbf{q}^{-4}\mathbf{t}^4 + \mathbf{a}^2(\mathbf{q}^2\mathbf{t}^3 + \mathbf{t}^2 + \mathbf{t}^3 + \mathbf{q}^{-2}\mathbf{t}^2 + \mathbf{q}^{-4}\mathbf{t} + \mathbf{q}^{-6}\mathbf{t}) + \mathbf{q}^{-6}\mathbf{t}^{-2} + \mathbf{q}^{-4}\mathbf{t}^{-1} + \mathbf{q}^{-2} + \mathbf{q}^{-2}\mathbf{t}^{-1} + 3 + \mathbf{q}^2\mathbf{t} +$$

$$+ \mathbf{q}^2 + \mathbf{q}^4\mathbf{t} + \mathbf{q}^6\mathbf{t}^2 + \mathbf{a}^{-2}(\mathbf{q}^{-2}\mathbf{t}^{-3} + \mathbf{t}^{-2} + \mathbf{t}^{-3} + \mathbf{q}^2\mathbf{t}^{-2} + \mathbf{q}^4\mathbf{t}^{-1} + \mathbf{q}^6\mathbf{t}^{-1}) + \mathbf{a}^{-4}\mathbf{q}^4\mathbf{t}^{-4}$$

$$\frac{{}^*\mathcal{P}^{4_1}_{[3]}(A|q,t)}{{}^*M_{[3]}(A\,|\,q,t)} \;=\; \mathbf{a}^6\mathbf{q}^{12}\mathbf{t}^{18} + \mathbf{a}^4(\mathbf{t}^7\mathbf{q}^2 + \mathbf{t}^8\mathbf{q}^4 + \mathbf{t}^9\mathbf{q}^4 + \mathbf{t}^{10}\mathbf{q}^6 + \mathbf{t}^{11}\mathbf{q}^6 + \mathbf{t}^{12}\mathbf{q}^8 + \mathbf{t}^{13}\mathbf{q}^{10} + \mathbf{t}^{15}\mathbf{q}^{12} + \mathbf{t}^{17}\mathbf{q}^{14}) +$$

$$+ \mathbf{a}^2(\mathbf{t}^{-2}\mathbf{q}^{-6} + \mathbf{t}^{-1}\mathbf{q}^{-4} + \mathbf{q}^{-4} + 2\mathbf{t}\mathbf{q}^{-2} + \mathbf{t}^2\mathbf{q}^{-2} + 2\mathbf{t}^2 + 2\mathbf{t}^3 + 3\mathbf{t}^4\mathbf{q}^2 + \mathbf{t}^5\mathbf{q}^2 + \mathbf{t}^5\mathbf{q}^4 + 4\mathbf{t}^6\mathbf{q}^4 + 2\mathbf{t}^7\mathbf{q}^6 + 2\mathbf{t}^8\mathbf{q}^6 + 2\mathbf{t}^9\mathbf{q}^8 +$$

$$+ \mathbf{t}^{10}\mathbf{q}^8 + \mathbf{t}^{10}\mathbf{q}^{10} + \mathbf{t}^{11}\mathbf{q}^{10} + \mathbf{t}^{12}\mathbf{q}^{12} + \mathbf{t}^{14}\mathbf{q}^{14}) + \mathbf{t}^{-9}\mathbf{q}^{-12} + \mathbf{t}^{-8}\mathbf{q}^{-10} + \mathbf{t}^{-7}\mathbf{q}^{-8} + \mathbf{t}^{-6}\mathbf{q}^{-8} + 3\mathbf{t}^{-5}\mathbf{q}^{-6} + \mathbf{t}^{-4}\mathbf{q}^{-6} + \mathbf{t}^{-4}\mathbf{q}^{-4} +$$

$$+ 4\mathbf{t}^{-3}\mathbf{q}^{-4} + 3\mathbf{t}^{-2}\mathbf{q}^{-2} + 3\mathbf{t}^{-1}\mathbf{q}^{-2} + \mathbf{t}^{-1} + 5 + \mathbf{t} + 3\mathbf{t}\mathbf{q}^2 + 3\mathbf{t}^2\mathbf{q}^2 + 4\mathbf{t}^3\mathbf{q}^4 + \mathbf{t}^4\mathbf{q}^4 + \mathbf{t}^4\mathbf{q}^6 + 3\mathbf{t}^5\mathbf{q}^6 + \mathbf{t}^6\mathbf{q}^8 + \mathbf{t}^7\mathbf{q}^8 + \mathbf{t}^8\mathbf{q}^{10} +$$

$$+ \mathbf{t}^9\mathbf{q}^{12} + \mathbf{a}^{-2}(\mathbf{t}^{-14}\mathbf{q}^{-14} + \mathbf{t}^{-12}\mathbf{q}^{-12} + \mathbf{t}^{-11}\mathbf{q}^{-10} + \mathbf{t}^{-10}\mathbf{q}^{-10} + \mathbf{t}^{-10}\mathbf{q}^{-8} + 2\mathbf{t}^{-9}\mathbf{q}^{-8} + 2\mathbf{t}^{-8}\mathbf{q}^{-6} + 2\mathbf{t}^{-7}\mathbf{q}^{-6} + 4\mathbf{t}^{-6}\mathbf{q}^{-4} +$$

$$+ \mathbf{t}^{-5}\mathbf{q}^{-4} + \mathbf{t}^{-5}\mathbf{q}^{-2} + 3\mathbf{t}^{-4}\mathbf{q}^{-2} + 2\mathbf{t}^{-3} + 2\mathbf{t}^{-2} + \mathbf{t}^{-2}\mathbf{q}^2 + 2\mathbf{t}^{-1}\mathbf{q}^2 + \mathbf{q}^4 + \mathbf{t}\mathbf{q}^4 + \mathbf{t}^2\mathbf{q}^6) +$$

$$+ \mathbf{a}^{-4}(\mathbf{t}^{-17}\mathbf{q}^{-14} + \mathbf{t}^{-15}\mathbf{q}^{-12} + \mathbf{t}^{-13}\mathbf{q}^{-10} + \mathbf{t}^{-12}\mathbf{q}^{-8} + \mathbf{t}^{-11}\mathbf{q}^{-6} + \mathbf{t}^{-10}\mathbf{q}^{-6} + \mathbf{t}^{-9}\mathbf{q}^{-4} + \mathbf{t}^{-8}\mathbf{q}^{-4}) + \mathbf{t}^{-7}\mathbf{q}^{-2}) + \mathbf{a}^{-6}\mathbf{t}^{-18}\mathbf{q}^{-12}$$

$$\dots$$

Note that these formulas for the superpolynomials, though natural, are not as well tested as those for the HOMFLY polynomials (4), simply because there is yet nothing to compare with. Still, they automatically include the famous formula [4] for the superpolynomial in the fundamental representation

$R = [1]$ and also seem to reproduce the recent answers [31] for the two-box cases, $R = [2]$ and $R = [11]$ (they are related by the symmetry transform (30)).[4] This is, however, a very small set of evidence as compared to the list in s.5 for HOMFLY polynomials, and further tests are very essential.

9   Difference equations for HOMFLY and superpolynomials

The colored Jones and refined Jones polynomials in representation $R = [p]$ are known to satisfy *linear* difference equations in the variable p [18, 22],[5] and it is natural to ask, if such equations exist for our superpolynomials (27). In fact, one expects a whole set of such equations to exist, a substitute of the set of the Virasoro constraints in matrix models [32]. This set defines the (quantum) spectral curve and serves as the starting point for the AMM/EO topological recursion [21]. So far most results in this direction are obtained for the colored Jones (super)polynomials, in particular, the recursion for the figure eight knot 4_1 is studied in [19] basing on $\mathcal{A}$-polynomial. The first equation of this kind for superpolynomials is just written in [22] for the case of the trefoil 3_1 (in fact, for the whole series of torus knots $(2, 2k+1)$), where it is derived from explicit expressions for the colored superpolynomials obtained by the "evolution" method of [9].

It is, therefore, natural to do the same for our explicit expressions (27). A new possibility here is to study the equations for arbitrary A. It turns out that the most natural linear equation is somewhat different from what one always considers: they involve variation not only in p, but also in A. Such a "basic" equation is of the first order, while for $A = q^N$ with fixed N one usually gets an Nth-order difference equation in p, which should be a corollary of the "basic" one. In the case of the superpolynomials it is also natural to expect that two "basic" equations should exist, associated with the horizontal and vertical shifts along the Young diagram R. However, only one is immediately seen from our formulas, because they are written only for the single-line diagrams $R = [p]$.[6] To write down the equation, we switch to still another notation, which makes explicit the dependence on p. Namely, instead of $\mathfrak{Z}_i(A) = \{Aq^{2(p-i)+1}\}\{At^{-1}\}$ we introduce $\mathfrak{Z}_{I|J} = \{Aq^I\}\{At^{-J}\}$, so that $\mathfrak{Z}_i = \mathfrak{Z}_{2p-2i+1|1}$. We also need a condensed notation for the action of the dilatation operator $\hat{D}_q : A \to qA$,

$$\mathfrak{Z}^{(s)}_{I|J}(A) = \hat{D}^s_q \mathfrak{Z}_{I|J}(A) = \mathfrak{Z}_{I|J}(q^s A) = \{Aq^{I+s}\}\{Aq^s t^{-J}\} \tag{33}$$

Then, the first few normalized superpolynomials $P_{[p]} = \frac{{}^*\mathcal{P}^{4_1}_{[p]}(A|q,t)}{{}^*M_{[p]}(A|q,t)}$ in (27) are:

$$P_{[0]} = 1,$$

$$P_{[1]} = 1 + \mathfrak{Z}_{1|1},$$

$$P_{[2]} = 1 + \left(\mathfrak{Z}_{3|1} + \mathfrak{Z}_{1|1}\right) + \mathfrak{Z}_{3|1}\mathfrak{Z}^{(1)}_{1|1},$$

$$P_{[3]} = 1 + \left(\mathfrak{Z}_{5|1} + \mathfrak{Z}_{3|1} + \mathfrak{Z}_{1|1}\right) + \left(\mathfrak{Z}_{5|1}\mathfrak{Z}^{(1)}_{3|1} + \mathfrak{Z}_{5|1}\mathfrak{Z}^{(1)}_{1|1} + \mathfrak{Z}_{3|1}\mathfrak{Z}^{(1)}_{1|1}\right) + \mathfrak{Z}_{5|1}\mathfrak{Z}^{(1)}_{3|1}\mathfrak{Z}^{(2)}_{1|1},$$

$$\ldots$$

so that

$$P_{[1]} - P_{[0]} = \mathfrak{Z}_{1|1} = \mathfrak{Z}_{1|1}P_{[0]},$$

$$P_{[2]} - P_{[1]} = \mathfrak{Z}_{3|1}\left(1 + \mathfrak{Z}^{(1)}_{1|1}\right) = \mathfrak{Z}_{3|1}P^{(1)}_{[1]},$$

$$P_{[3]} - P_{[2]} = \mathfrak{Z}_{5|1}\left(1 + \mathfrak{Z}^{(1)}_{1|1} + \mathfrak{Z}^{(1)}_{3|1} + \mathfrak{Z}^{(1)}_{3|1}\mathfrak{Z}^{(2)}_{1|1}\right) = \mathfrak{Z}_{5|1}P^{(1)}_{[2]},$$

$$\ldots$$

---

[4] In fact, in [31] a different grading was used, still it looks plausible that those formulas are consistent with ours.

[5] An even more interesting question is if the non-linear equations also exist, generalizing *bi*-linear integrable Hirota-like equations. In this direction only some preliminary results for the torus knots are available, see [14].

[6] Of course, another equation, in t-direction, immediately follows for the set $R = [1^p]$ in (31), but in order to see both equations at once, one should look at the answer for generic R.

and in general

$$P_{[p+1]}(A) - P_{[p]}(A) = \mathfrak{Z}_{2p+1|1}(A)P_{[p]}^{(1)}(A) = \{Aq^{2p+1}\}\{At^{-1}\}P_{[p]}(qA) \tag{34}$$

Here t is just a parameter, and one obtains an equation for the normalized HOMFLY polynomials just by putting $t = q$:

$$h(A, p+1) - h(A, p) - \left(A^2 q^{2p} + \frac{1}{A^2 q^{2p}} - q^{2p+2} - \frac{1}{q^{2p+2}}\right)h(qA, p) = 0 \tag{35}$$

for $h(A, p) = {}^* \mathcal{H}_{[p]}^{4_1}(A)/{}^* S_{[p]}(A)$. This equation can be rewritten in terms of "the quantum $\mathcal{A}$-polynomial",

$$\hat{\mathcal{A}}\left(\frac{{}^* \mathcal{H}_{[p]}^{4_1}(A \,|\, q)}{{}^* S_{[p]}(A \,|\, q)}\right) = 0, \qquad \hat{\mathcal{A}} \equiv \hat{l} - 1 - \left(A^2 \hat{m}^2 + \frac{1}{A^2 \hat{m}^2} - q^2 \hat{m}^2 - \frac{1}{q^2 \hat{m}^2}\right)\hat{D} \tag{36}$$

where, as usual, $\hat{l}f(A, p) = f(A, p+1)$ and $\hat{m}f(A, p) = q^p f(A, p)$. As already mentioned, this equation differs from the standard Jones polynomial equation as it involves two variables: p (the representation) and A (the group), on the other hand, it is the first order difference equation, while the Jones polynomial equation is the second order difference equation in one variable p. One may expect difference equations of higher orders in this one variable in the case of $A = q^N$ with $N > 2$, however, for generic A it is unclear if an equation of such a type exists at all.

10 On possible generalizations

There are two obvious directions to further generalize and explore the result (4):

- from $R = [p]$ to arbitrary R (arbitrary Young diagram $R = \{p_1 \geq p_2 \geq \ldots \geq 0\}$),

- from $\mathcal{K} = (1, -1|1, -1)$ to entire series of 3-strand knots $\mathcal{K} = (1, -1)^n = (1, -1|1, -1|\ldots|1, -1)$, a simple generalization of the torus knot family $\mathcal{K} = [3, n] = (1, 1)^n$ (these are knots for n, indivisible by $m = 3$ and 3-component links otherwise),

- one can also convert to the character decomposed form of the formula and find for this family an analogue of the Rosso-Jones formula [33] for the torus knots (this was done in [12] for the fundamental representation only, so there was no chance of formulating a counterpart of the Adams rule).

As concerns the first item in this list, eqs.(4) and (8) are expected to extended naturally to a full formula. Like in the case of superpolynomials, more suited for generalization is not (4) itself, but its reformulation (26), which is just a sum of contributions from al subsets of boxes in the Young diagram R taken with unit weights. However, though easily conjectured, the formulas for arbitrary R can not be sufficiently tested at the moment due to the lack of any alternative calculations. To avoid mixing reliable formulas with speculations, we discuss generic representations elsewhere.

Acknowledgements

References

[1] J.W.Alexander, Trans.Amer.Math.Soc. **30** (2) (1928) 275?306;
J.H.Conway, Algebraic Properties, In: John Leech (ed.), *Computational Problems in Abstract Algebra*, Proc. Conf. Oxford, 1967, Pergamon Press, Oxford-New York, 329-358, 1970;
V.F.R.Jones, Invent.Math. **72** (1983) 1 Bull.AMS **12** (1985) 103Ann.Math. **126** (1987) 335;
L.Kauffman,Topology **26** (1987) 395

[2] P.Freyd, D.Yetter, J.Hoste, W.B.R.Lickorish, K.Millet, A.Ocneanu, Bull. AMS. **12** (1985) 239
J.H.Przytycki and K.P.Traczyk, Kobe J. Math. **4** (1987) 115-139

[3] S.Gukov, A.Schwarz and C.Vafa, Lett.Math.Phys. **74** (2005) 53-74, arXiv:hep-th/0412243;
E.Gorsky, arXiv:1003.0916;
N.Carqueville and D.Murfet, arXiv:1108.1081;
I.Cherednik, arXiv:1111.6195;
A.Oblomkov, J.Rasmussen and V.Shende, arXiv:1201.2115 (with an Appendix by Eugene Gorsky)

[4] N.M.Dunfield, S.Gukov and J.Rasmussen, Experimental Math. 15 (2006) 129-159, math/0505662

[5] M.Aganagic and Sh.Shakirov, arXiv: 1105.5117

[6] S.-S.Chern and J.Simons, Ann.Math. **99** (1974) 48-69;
E.Witten, Comm.Math.Phys. **121** (1989) 351;
G.Moore and N.Seiberg, Phys.Lett. **B220** (1989) 422;
V.Fock and Ya.I.Kogan, Mod.Phys.Lett. **A5** (1990) 1365-1372;
R.Gopakumar and C.Vafa, Adv.Theor.Math.Phys. **3** (1999) 1415-1443, hep-th/9811131

[7] E.Guadagnini, M.Martellini and M.Mintchev, In Clausthal 1989, Proceedings, Quantum groups, 307-317; Phys.Lett. B235 (1990) 275;
N.Yu.Reshetikhin and V.G.Turaev, Comm. Math. Phys. **127** (1990) 1-26

[8] H.Morton and S.Lukac, J. Knot Theory and Its Ramifications, **12** (2003) 395, math.GT/0108011

[9] P.Dunin-Barkowski, A.Mironov, A.Morozov, A.Sleptsov and A.Smirnov, arXiv:1106.4305 v2

[10] A.Mironov, A.Morozov, Sh.Shakirov and A.Sleptsov, arXiv:1201.3339

[11] A.Morozov, arXiv:1201.4595

[12] A.Mironov, A.Morozov and And.Morozov, JHEP **03** (2012) 034, arXiv:1112.2654

[13] A.Mironov, A.Morozov and Sh.Shakirov, arXiv:1203.0667

[14] A.Mironov, A.Morozov and And.Morozov, arXiv:1112.5754

[15] R.K.Kaul and T.R.Govindarajan, Nucl.Phys. **B380** (1992) 293-336, hep-th/9111063;
P.Ramadevi, T.R.Govindarajan and R.K.Kaul, Nucl.Phys. **B402** (1993) 548-566, hep-th/9212110; Nucl.Phys. **B422** (1994) 291-306, hep-th/9312215;
Zodinmawia and P.Ramadevi, arXiv:1107.3918

[16] P.Ramadevi and T.Sarkar, Nucl.Phys. **B600** (2001) 487-511, hep-th/0009188

[17] K.Kawagoe, in: *Intellegence of Low Dimensional Topolgy 2006*, Eds.J.Scott Carter at al., World Scientific Publishing Co., 2007, pp.143-150

[18] R.Gelca, Math. Proc. Cambridge Philos. Soc. **133** (2002) 311-323, math/0004158;
R.Gelca and J.Sain, J. Knot Theory Ramifications, **12** (2003) 187-201, math/0201100;
S.Gukov, Commun.Math.Phys. **255** (2005) 577-627, hep-th/0306165;
S.Garoufalidis and T.Le, Geometry and Topology, **9** (2005) 1253-1293, math/0309214

[19] R.Dijkgraaf, H.Fuji and M.Manabe, Nucl.Phys. **B849** (2011) 166-211, arXiv:1010.4542

[20] S.Gukov and P.Sulkowski, arXiv:1108.0002

[21] A.Alexandrov, A.Mironov and A.Morozov, Int.J.Mod.Phys. **A19** (2004) 4127, hep-th/0310113; Teor.Mat.Fiz. **150** (2007) 179-192, hep-th/0605171; Physica **D235** (2007) 126-167, hep-th/0608228; JHEP **12** (2009) 053, arXiv:0906.3305;
A.Alexandrov, A.Mironov, A.Morozov, P.Putrov, Int.J.Mod.Phys. **A24** (2009) 4939-4998, arXiv:0811.2825;
B.Eynard, JHEP **0411** (2004) 031, hep-th/0407261;
L.Chekhov and B.Eynard, JHEP **0603** (2006) 014, hep-th/0504116; JHEP **0612** (2006) 026, math-ph/0604014;
N.Orantin, arXiv:0808.0635

[22] H.Fuji, S.Gukov and P.Sulkowski (with an appendix by Hidetoshi Awata), arXiv:1203.2182

[23] H.Itoyama, A.Mironov, A.Morozov and And.Morozov, *HOMFLY polynomial for all 3-strand braids in symmetric representation*, to appear

[24] Knot Atlas at http://katlas.org/wiki/Main_Page (by D.Bar-Natan)

[25] H.Itoyama, A.Mironov, A.Morozov and And.Morozov, *Colored HOMFLY polynomials for all 3-strand braids*, to appear

[26] H.Itoyama, A.Mironov, A.Morozov and And.Morozov, *"Special" polinomials for 3-strand braids*, to appear

[27] N.Vilenkin and A.Klymik, *Representation of Lie groups and Special Functions*, Volume 3, Mathematics and its applications, Kluwer academic publisher, 1993

[28] A.N.Kirillov and N.Yu.Reshetikhin, *Representations of the Algebra Uq(2), q-Orthogonal Polynomials and Invariants of Links*, Preprint, 1988;
L.Alvarez-Gaune, C.Gomez and S.Sierra, Phys.Lett. **B220** (1989) 142

[29] R.M.Kashaev, Mod.Phys.Lett. **A9** (1994) 3757-3768, hep-th/9411147; q-alg/9504020; Lett.Math.Phys. **39** (1997) 269-265, q-alg/9601025

[30] H.Ooguri and C.Vafa, Nucl.Phys. **B577** (2000) 419-438, hep-th/9912123;
J.Labastida, M.Marino, Comm.Math.Phys. **217** (2001) 423-449, hep-th/0004196; math/010418;
J.M.F.Labastida, M.Marino and C.Vafa, JHEP **0011** (2000) 007, hep-th/0010102;
M.Marino and C.Vafa, arXiv:hep-th/0108064

[31] S.Gukov and M.Stosic, arXiv:1112.0030

[32] A.Morozov, Phys.Usp.(UFN) **35** (1992) 671-714; **37** (1994) 1, hep-th/9303139; hep-th/9502091;
A.Mironov, Int.J.Mod.Phys. **A9** (1994) 4355, hep-th/9312212; Phys.Part.Nucl. **33** (2002) 537

[33] M.Rosso and V.F.R.Jones, J. Knot Theory Ramifications, **2** (1993) 97-112
X.-S.Lin and H.Zheng, Trans. Amer. Math. Soc. 362 (2010) 1-18 math/0601267;
S.Stevan, Annales Henri Poincaré 11 (2010) 1201-1224, arXiv:1003.2861;
A.Brini, B.Eynard and M.Mariño, arXiv:1105.2012